Introduction

Mathematical Work

Over the last couple of decades an increasing number of academic researchers across a wide range of fields including philosophy, cognitive science, archaeology, anthropology, and psychology have argued that many of the processes we call thought do not happen inside the head but are actually carried out in the environment, that thinking mainly happens beyond the skin in ways analogous to how cooking begins the process of digestion outside the body.[1] This seems especially true of mathematics. If asked to multiply 2,437 and 457,334 very few of us, including professional mathematicians, would do the calculation in our heads. Instead, we reach for pencil and paper and perform a written algorithm (or better still, get our phones to do it). According to this "extended mind thesis," complex thinking, of the kind that mathematics requires, is pursued through the material environment with what has been called *mindware*.[2]

Historians have long used the material traces of mathematical work to write their histories. But less attention has been paid to the material form of those traces than to their symbolism. That is, of course, entirely natural. The importance of archival materials for exploring the history of mathematics is due, after all, to the symbols, writings, and drawings they contain. But in recent years changes in reading and writing technologies have invited a different perspective. When the print and paper book began to face the challenge of the electronic word in the shape of CD-ROMs and ereaders, the book became an object of considerable historical interest. As a result, our understanding of how texts were made and used was transformed. The computer also changed mathematical work, challenging the nature of mathematical proof, for example.[3] The graphing calculator is now well established in high school calculus classes and its use has made pencil-and-graph-paper technologies appear more and more like the dusty remnants of a vanishing past. When it is possible to pull up, zoom in, and whiz through a whole series of mathematical curves on a calculator screen, sketching a function and marking its intercepts on a piece of paper now seems like a foundational skill useful only as groundwork for a more exciting and accessible set of skills that will employ electronic tools. From the high school classroom to the research university, mathematics can now be explored in virtual reality and, as Brian Rotman and Ian Hacking have recognized, that has made it look less like an ideal science or art of necessary truth, and more like an experimental practice.[4]

These more recent changes should not lead us to think that things have only just now begun to have a cognitive life. Archaeologist Lambros Malafouris has argued that hominids began thinking through things as soon as they began to make tools, indeed that tools made humans just as much as humans made tools.[5] The environments we think with and through do not, therefore, have to involve complicated machines or fancy electronic devices. Philosopher and cognitive scientist Andy Clark has described how a community of people with Alzheimer's living in St. Louis built an environment full of cognitive props and tools out of sticky notes, photographs, and open storage strategies. The mindware they created reminded them of tasks that had to be completed, of names and relationships to family and friends, and where important items such as pots, pans, and checkbooks were to be found. Members of this community performed very badly on cognitive tests. Yet, with the aid of a cognitive scaffolding made out of sticky notes, pictures, and cupboards without doors, they lived independent lives that went way beyond that which their measured cognitive capacities might have been expected to support.[6] Science studies scholar Hélène Mialet has also shown how a social and material environment can enable someone as physically challenged as physicist Stephen Hawking to perform theoretical work at the very highest level.[7]

Some of us might be tempted to dismiss these examples because they involve humans with cognitive and physical impairments that make them different. But the approach I take in this book rests on the conviction that what these stories make visible is true of us all. Any mindful work, whether that of a builder, a painter, a natural philosopher, or a mathematician must be performed through and with their environment.[8] The subjects of this book, eighteenth- and nineteenth-century British mathematicians, worked in a rapidly changing social and material environment that played an active role in the cognitive work they performed. As one of those subjects, nineteenth-century natural philosopher William Thomson, remarked, "The expenditure of chalk is often a saving of brains."[9] This book is a historical investigation of the way mathematical work and the highly abstract and ideal concepts it employs change with the material environment.

Beyond the Hand

Recent work in the history and critical theory of the mathematical sciences has focused attention on theoretical work as an embodied and craft practice. Andrew Warwick has drawn attention to the material culture of mathematics as performance on paper rather than in the head, a skilled activity that developed with innovations in mathematical technique.[10] David Kaiser and others have explored how teaching makes mathematicians into a highly disciplined community committed to particular formal practices.[11] At the same time, increasing recognition of the role played by theoretical technology such as pens, paper, desks, chalk, and blackboard has also called into question the gap between material and cognitive practices, between brainwork and handwork.[12]

These approaches to mathematical or theoretical work have led to talk of the "practice-ladenness of theory," an approach to mathematical work that draws attention away from the properties of number and toward mathematics as a human activity that involves a variety of material practices.[13] But at the same time, what is meant by those terms has also introduced an element of ambiguity into an important and deceptively complex question: what is a tool? Warwick, for example, has used the term *theoretical technology* to describe what appears most straightforwardly as the pens, paper, books, tables, slide rules, or other kinds of calculating machines or devices that a mathematician might use. But for Ursula Klein and David Kaiser the term *paper tools* also refers to the manipulation of symbols (or diagrams) on paper, which seems to imply that the practice is itself a tool rather than a way of doing things that employs tools.[14] Klein makes the equivocation explicit: "Even though terms like conceptual or mathematical *tools*, *toolbox of science*, *theoretical technology*, and so on are sometimes used in a strictly metaphoric sense and hence do not attribute a material dimension to these entities, they have nevertheless narrowed the former gulf between manual and cognitive practices and called into question the dichotomy between hand and mind, seeing and thinking."[15] The problem is not simply to show the continuity between tools and the conceptual manipulation of symbols but also to account for the discontinuities between tool use and theoretical practice that shift according to how both tools and symbols are employed.

Tools shape practice but practice also makes tools. Pen and paper become a mathematician's experimental apparatus if they are used in particular ways. The manipulations of symbols on paper might be employed as a kind of paper tool, perhaps as a kind of algorithmic procedure that is useful for a number of applications. But it might also be an exploratory practice, an investigation of the virtual space the manipulations generate, a way of thinking. These kinds of considerations suggest a dynamic relationship between ways of doing, the tools those ways employ, and the practitioners and the results that are coproduced with them.

A theoretical worker will use many different kinds of things, not just pen and paper but also books, notebooks, journals, mathematical instruments, and mundane objects to perform their work. In the nineteenth century that work might include not only the manipulation of symbols but also sketching diagrams or visualizing and analyzing problems. Even the seemingly simplest instance, a mathematician manipulating symbols on paper with a pen, is an example of a cognitive practice that must include not only what goes on in the mathematician's brain and the manipulations of symbols but also other kinds of materials—books, libraries, journals, letters, and other mathematicians (among other things and persons)—that are distributed across different places and even other times.[16] As philosopher of science Joseph Rouse has commented, practices are always interconnected in ways that mean they cannot be spatially or temporally bounded.[17] Yet, at the same time, as Warwick and Kaiser have shown, if we are to understand mathematical work as a craft practice it must also be performed in a particular time and place.[18]

The approach I take here, then, is to see mathematics as a local but unbounded practice—local in the sense that a mathematician works in a particular place, but unbounded in the sense that its practice involves cognitive work that is distributed across times and places. That will require seeing features of the mathematician's physical surroundings as active participants in the process of their thought. I therefore place Victorian mathematicians in a dynamically changing material environment that can be imagined as something like anthropologist Tim Ingold's conception of the taskscape.[19] Ingold's taskscape is a musical image of an environment that is constantly changing at different registers simultaneously, like melodies interweaving in a piece of polyphonic music. The dynamics of the taskscape are shaped by human activity, tasks car-

Figure I.1: Ford Madox Brown, *Work* (1852–1865). Image courtesy of Manchester Art Gallery.

ried out by skilled agents that can include large projects like building a library as well as smaller ones like preserving letters in a letter-book. The tasks contribute to how the landscape changes so that the landscape is not merely a static material backdrop against which these tasks are undertaken; it is dynamic and animated, changing simultaneously with the temporal melody lines of the taskscape. It is not that the taskscape is cultural and the landscape natural, it is rather that the taskscape *is* the embodied landscape.

The Temporality of the Victorian Landscape

To illustrate his temporal landscape, Ingold invites his reader to step inside a painting, Pieter Bruegel's *The Harvesters*.[20] The taskscape that we will encounter in the following pages is very different from the one Ingold describes and I therefore want to invite the reader into a picture more appropriate for the largely urban Victorian environment with which this book is concerned: Pre-Raphaelite painter Ford Madox Brown's *Work*.[21]

The focus of Brown's *Work* is a group of laborers digging a hole on The Mount, a Hampstead street in mid-nineteenth-century London. At the center of the picture is a group of laborers around which a hexagonal mass of people spirals outward. One of the workers, standing just to the left of the diggers, is taking a break and drinking a beer. The beer drinker is standing next to a beer seller calling out his wares. Just below them some children and a couple of small dogs are playing while a very young woman, barely more than a child herself, is vainly trying to keep them under control. To the workers' right, a rather disreputable-looking character, who clearly lives a marginal existence supported by selling flowers and plants, walks ahead of a file of people. Behind him are two respectable-looking Victorian women, making their way around the hole the laborers are digging and toward the viewer.[22] At the very back, behind the workers and in shadow, are a wealthy couple on horseback. To the right, just below the road, some agricultural workers sleep in the shade of a tree, a couple feed a baby, and an individual in a stovepipe hat enjoys a pipe in the shade. In the far distance are a number of figures, including some with placards campaigning for the election of a member of Parliament in a road leading up to what appears to be a high street. In the foreground on the far right of the picture are cultural critic Thomas Carlyle and theologian Frederick Denison Maurice, two Victorian intellectuals whom Brown describes as *brain workers*, leaning against a fence. The heroic male figures whose hard work is the focus of Brown's painting provide a center of gravity that seems to sustain and repair the social and material fabric of the Victorian life that orbits around them.

Some of the mathematicians and natural philosophers with whom this book is concerned—Augustus De Morgan, for example—can easily be imagined walking in this landscape. But if we saw De Morgan on the Mount in Hampstead, the setting for Brown's picture, he would appear to us as having more in common with the brain workers—Carlyle, Maurice, or even Brown himself—than with the workers at the center of the picture. Nonetheless, in order to try to communicate a better appreciation of the taskscape that will be explored in this book I will offer an unlikely analogy, one between mathematical work and the hard, physical labor that occupies the workers digging up the road in Brown's painting. The workers in the picture are not using pens but they do hold shovels, which appear as extensions of their bodies. If we look closely enough, those bodies also appear to extend into the landscape itself. Indeed, one worker is so far inside the hole that all we can see of him is a hand and shovel. The three workers shoveling soil out of the hole appear as practically continuous with each other and their environment in the tasks they are performing. Through their shovels they are prosthetically extended into the performance of digging the hole, and through their cooperation they act together, as one organism. But these continuities are not permanent. To the workers' left, the beer seller from the nearby Prince of Wales public house is tempting them to call a halt to their digging, put their shovels down, and take time to buy a drink and to refresh themselves. If they do that they will change; they will become drinkers, not workers. Indeed, as already noted, one of the crew, depicted standing behind the beer seller, has already done so.[23]

One way in which the laborers' work might appear to be a long way from that of a theoretical worker is that the laborers' tasks require them to work together, in apparent contrast to a mathematician, who seems perfectly able to work alone. However, this is not the case.[24] The physicality of the bodies in Brown's painting are a wonderful illustration of how those engaged in the performance of a task must attend to others performing related tasks. Mathematicians may not necessarily be physically present with one another when they work, but they must still attend to one another even if that attention is mediated by communication technologies such as print or letters. Furthermore, to dig a hole in London requires considerable knowledge about the pipes, wires, or tunnels that lie underground. The laborers' work in Victorian London is embodied in the Hampstead landscape not only through the metal plates still visible in the road today and through which the pipes or tunnels they were digging can still be accessed but also in records that have to be consulted whenever any kind of digging work is undertaken in twenty-first-century London. Similarly, the mathematical equations, diagrams, tables, or pictures that fill the manuscript papers, published articles, or other printed materials with which this book is concerned are the material traces of theoretical work. Through the tools that they use—the pens, letters, and books, as well as assemblings of things such as libraries, museum collections, and societies—mathematicians are extended into their social and material environment and so should not be supposed to be working in isolation.

Lastly, if we imagine ourselves watching the workers from inside the picture we can also get a sense of how the different temporalities of their environment play out over one another. Digging the holes, laying the pipes, and then repairing the ground are tasks that happen over time, perhaps occurring over a few weeks or months and then coming to an end. But through those activities the workers will leave behind marks in the landscape (most obviously the metal access plate into the tunnel they are digging) that embody their movements in the taskscape in a way that has endured. But that is hardly the only change that has occurred. In Brown's picture the road is not yet asphalted, although it will be at some time in the future. Some of the houses in the background behind the two mounted figures have hardly changed at all, while other parts of the landscape, most noticeably the road leading back and to the right in Brown's picture, have been transformed by a zebra crossing, modern traffic, and buildings that make them almost unrecognizable today. Other kinds of movement, such as the growth of bushes and plants, also make change. Each of these myriad changes has its own temporality; some are visible, as when a building construction takes place, others are imperceptible, like the growth of a mature tree. The landscape is dynamic and the tasks performed in it participate in and contribute to the changes through which it is constantly transformed.

Mathematical work, I argue, is no different. The Victorian mathematicians and theoretical workers in this book act in ways that are part and parcel of the ways in which their world is transforming itself, a world in which everything is in movement, animated, and changing simultaneously through a number of temporalities. There is, of course, no way that anywhere near a full account of that environment can be achieved in the limited space available here, but a partial description of particular changes in the late Georgian and early Victorian British taskscape will show how nineteenth-century mixed mathematicians used, made, and remade the ways of thinking the changes in their environment afforded.

The Structure of the Book

In the period I cover in this book—from the end of the early modern period into the beginning of the modern era—there was no clear distinction between mathematics and physics and many of those engaged in what I will call *theoretical work* will look very different from the figure of the twenty-first-century mathematician. Some, but by no means all of them, worked in universities at some point in their careers. William Thomson was a professor of natural philosophy at Glasgow University. One of his friends, James Clerk Maxwell, was the first Cavendish Professor of Experimental Physics at Cambridge University. Another, Peter Guthrie Tait, was mathematics professor at Queen's College Belfast and then professor of natural philosophy at Edinburgh University. None of them would have described themselves as theoretical physicists, a kind of theoretical worker that only emerged toward the end of the nineteenth century.

Significant theoretical work also took place outside the universities. George Green published the work for which he is now best remembered when he was working as a miller. George Boole wrote his groundbreaking *Mathematical Analysis of Logic* when he was a schoolteacher. There were attempts to construct and maintain disciplinary boundaries, to define the fields and subfields of mathematics and physics in the nineteenth century, but they were never clearly established and maintained in the historical periods with which this book is concerned.[25] Therefore, I do not try to fit the particular communities of mathematicians considered here into modern disciplinary categories. Instead, the book is organized around the idea that theoretical workers, like the laborers digging the hole in Hampstead, are made through and with their tasks and the tools that they use.

Each of the chapters explores a thing, or assembling of things, that late Georgian and early Victorian mathematicians needed to do their work: textbooks, journals, museums, libraries, diagrams, notebooks, and letters. The items are not supposed to be a complete list—it bears repeating that mathematicians use a wide variety of things in the performance of their work—but have been chosen to help facilitate my historical narrative. Some might expect a work on the material culture of mathematics to feature well known nineteenth-century objects such as Charles Babbage's difference engine or Thomas De Colmar's arithmometer. They will, however, be disappointed. Although it is true that in the second half of the nineteenth-century aids to calculation such as mechanical calculators were beginning to find their way not only into the actuary's office but also into the theoretical worker's toolkit, I am more concerned with an earlier period and a different material environment.[26] Babbage is discussed here, but I am much more interested in his manuscript "Essays in the Philosophy of Analysis," which circulated in the paper world of early nineteenth-century Cambridge, than in his difference engine, which never went beyond the prototype stage and so never became a prominent feature in the late Georgian or early Victorian theoretical worker's taskscape. It was not calculating machines, but glue, scissors, pen, and ink, that Tait used when he struggled to master Hamilton's quaternion mathematics in 1858.

The first two parts of the book are about the becoming of what I would argue was a world more familiar to the nineteenth-century mathematician, that of textbooks, journals, museums, and libraries. Part three focuses on how leading Victorian theoretical workers Maxwell, Thomson, Tait, and Hamilton performed the work their taskscape afforded. The book describes how British mathematical work changed over the eighteenth and into the middle of the nineteenth century. Some of the changes highlighted in the stories told here include how algebra superseded the classical authority of geometry as the language of British mathematics; how Leibnizian notation became the language of the integral and differential calculus; and how impossible quantities such as the square root of negative one were used to make a new mathematical object, the vector, which is a quantity that has magnitude and direction. At the same time, and not coincidentally, theoretical workers also began to think through and with a new material environment of journals, modern textbooks, libraries, the Royal Mail, and even new kinds of objects such as batteries.

While older histories once focused on the considerable conceptual changes in British mathematics over this period, more recent accounts have focused on the importance of changes in reading and writing practices especially as they shaped, and were shaped by, the increasingly competitive Cambridge University Senate House Exam, or Tripos. Work by Jonathan Topham, Gert Schubring, and Sloan Evans Despeaux, among others, has drawn attention to textbooks and journals and, in so doing, brought mathematics into the increasing orbit of the history of the book and of reading and writing practices. I have sought to pursue this direction in the chapters on textbooks, journals, notebooks, libraries, and letters.

In Part I, consisting of chapters 1, "Textbook in the Marketplace," and 2, "Fences, Diaries, and Mathematics Journals," I take up the question of how the early twenty-first-century understanding of what counts as a mathematics textbook or journal can be too easily projected onto the past, creating an illusion of continuity that obscures a more complicated and interesting story. Closer attention to the material forms of the mathematical textbook and the journal reveal discontinuities in mathematical text making and how different kinds of publication are coproduced with the communities they serve. Mathematics communities are made through the collaboration of a variety of workers, including printers, publishers, and booksellers. Looking at the production of mathematical texts over a longer period, from the beginning of the eighteenth century into the nineteenth shows not only how mathematics communities make themselves but also how they are entangled with a wider taskscape, a material environment that can change radically and eventually fail to sustain the patterns of activity and circumstances that once made a particular community's practices viable.

Mathematicians think with pens, paper, journals, and books but also with other kinds of things, including instruments. There are good reasons to think, for example, that eighteenth-century mathematicians thought much more with the latter than the former. Better appreciation of the way mathematicians think with their environments suggests a different, closer relationship between theoretical and experimental work than has previously been supposed. Twentieth-century historian of science Thomas Kuhn's binary opposition between what he once called the mathematical and experimental traditions now appears untenable.[27] More recently, historians such as John Pickstone have argued for different ways of knowing that broadly break down into three basic categories: *natural history*, or describing and classifying; *analyzing*, breaking things down into various kinds of elements; and *experimenting* to control phenomena and create new entities.[28] Pickstone's ways of knowing allow a greater specificity about experimental practice while at the same time acknowledging the different kinds of practice involved in laboratory work. As a result, historians of science now better appreciate the role of collecting and the museum in the development of nineteenth-century laboratory sciences.[29]

Part II, consisting of chapters 3, "Cambridge Museological Science and the Making of English Algebra" and 4, "The Mathematician's Library," offers specific examples of the way in which mathematical work involves collecting practices and the way that English algebra, like the laboratory, grew up in the shadow of the museum. These stories, unlike those about textbooks and journals, are explored at a different historical timescale. While the chapters on textbooks and journals look back into the early modern period, the museum and library stories are concerned with specifically nineteenth-century practices and institutions.

Part III, which comprises the last three chapters, "Romantic Space and Imaginary Numbers," "William Thomson's Notebooks," and "Kites and Letters," looks more closely at specific theoretical workers and the kinds of practices that the changes in the nineteenth-century material environment afforded. The first of these, chapter 5, explores the overlap between experimental and theoretical work through what I have called Maxwell's diagrammatic practice. Maxwell's electromagnetic theory is one of the most celebrated achievements of nineteenth-century natural philosophy, and much has been written about his use of mechanical models in his development of what physicists now call *field theory*. Physicists and applied mathematicians sometimes marvel at the appropriateness of mathematics for describing nature, describing it as a "wonderful gift which we neither understand nor deserve" rather than the result of very hard work.[30] But is that correspondence really as magical as it now appears? As the archaeologist Ian Hodder has remarked, "we tend to forget the history of things" in part because the strings of entanglement between them are so long and complex.[31] In my chapter on Maxwell, I pick through a tangle of relationships between the production of nineteenth-century space, exploratory science, and the history of complex numbers to reveal a richer if less wonderful relationship between mathematics and natural philosophy.

In the last two chapters I focus on the material practice of nineteenth-century theoretical work through two extraordinary archive materials. Maxwell's older colleague, William Thomson, later Lord Kelvin, was famous for his notebooks, and for chapter 6 I use the 180 "Kelvin Notebooks" held in the Cambridge University Library to look more closely at how Victorian theoretical work was explored though both reading and writing practices, and especially how Thomson's brainwork extended beyond his body in a highly creative and dynamic relationship with his material environment. Exploring that dynamic relationship also offers some answers to the complex problem of the relationship between a theoretical worker and their historical context.

In the last chapter, the archival material is a letter-book made by physicist and mathematician Peter Guthrie Tait when he was in correspondence with the Anglo-Irish astronomer and mathematician William Rowan Hamilton. Unlike Thomson's notebooks, which cover a period roughly equal to Queen Victoria's reign, the most significant part of the letter-book records a moment of intense exchange between Tait and Hamilton that lasted only a few months. In that period, Tait and Hamilton thought together about a new kind of mathematics that Hamilton had invented, called quaternions, that would subsequently play an important role in the developments of both electromagnetic theory and vector calculus. The making of Tait's letter-book helped achieve a correspondence between two leading Victorian mathematicians working together in different and distant geographical locations. By following how their letters mounded up and Tait's letter-book began to grow we will see how Hamilton and Tait performed their mathematics together through the medium of the Victorian post in something like the way two musicians play together.

This book is concerned with how humans and things become entangled in theoretical work. The types of things with which it is concerned—tools such as pens, papers, corks, and fluids; printed materials such as textbooks and journals; assemblings of things in museums or libraries; notebooks; and letters—afforded some of the most important achievements in nineteenth-century British mathematics. But even in the choice of these actors and things it must not be forgotten that what counts as a historically significant story for any particular historian, including this one, must necessarily be only a small part of a much larger set of possible stories. If the approach I take here has any value it follows that any historical narrative must be imposed on a hugely complex and, for the historian's purposes, almost infinitely extendable taskscape. It is not that there cannot be any single correct narrative; it is that any attempt at capturing the messy complexity of the past must involve a field of narratives exploring different temporalities, historical actors, and social and material environments. Any history must therefore be incomplete. But I hope there is enough material in the stories told here to give the reader some idea of how nineteenth-century British theoretical workers thought through and with their physical environment to make mathematics.